MATH 504: CHAPTER 1

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1. A little bit about the philosophy of mathematics and the Universe of Sets

- Formalism vs Platonism. We will follow a Platonistic approach using formalistic methods and call the mathematical universe V.
- The naïve approach, the "definition of a set" and Russel's paradox vs the axiomatic approach. We will follow the axiomatic approach.
- Set theory is formulated in first-order logic with equality (together with all the logical axioms associated) in the language $\mathcal{L} = \{\in\}$ where " $\in$ " is a binary relation. We think of V as a model of in this language, satisfying the ZFC (or ZF) axioms that we will list in the next subsection. Through the axioms, we will get some sort of idea how V looks like.
- Working in V is only a technical issue since we will always prove from the axioms that the objects we are interested in exist, and therefore the model which we start with does not matter.

2. The theory of sets

2.1. Existence, Extensionality and Comprehension.

Ax0.(Existence) $\exists x.x = x$

From the set existence we can only prove that the universe is non-empty, for example $\{\emptyset\}$ is a model for this.

Ax1.(Extensionality) $\forall x \forall y((\forall z, z \in x \leftrightarrow x \in y) \rightarrow x = y)$

The axiom of extensionality is not contributing to the existence of new sets. It is used usuality to prove uniqueness.

Example 2.1. Let us claim that from extensionality, if there is a set x such that $\forall z.z \notin x$ then x is unique.

Proof. Suppose that x_1, x_2both satisfy that for all $\forall z.z \notin x_i$ $(i = 1, 2)$, then the antecedent $\forall z.z \in x_1 \leftrightarrow z \in x_2$ is satisfied and therefore $x_1 = x_2$. □

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For every formula ϕ in the language of set theory, such that y is not free in ϕ we have the following axiom scheme which is the universal closure of the following:

$$\textbf{Ax3.(Comprehension scheme)}\ \forall x \exists y \forall z . z \in y \leftrightarrow z \in x \wedge \phi$$

The intention of this axiom is to define sets of the form $\{z \in x \mid \phi\}$, and the resulting set is the variable y appearing in the axiom. The universal closure of the formula ensures that we can use parameters in our definition of a set. This axiom guarantees the existence of a unique empty set since:

(1) By Ax1, there is some x.
(2) By Ax3 applied to the formula $z \notin x$ we get a set y such that $\forall z.z \notin y$ (since $z \in x \wedge z \notin x$ is always false).
(3) By Ax2, as we have seen, such a set y is unique.

Now since this set is unique we reserve a special symbol for it

Definition 2.2. Assume Ax1,Ax2,Ax3. Let $\emptyset$ denote the unique set y such that $\forall z.z \notin y$.

Also, using comprehension, we get that Russell's paradox is simply a theorem:

Theorem 2.3. $\neg x.\forall z.z \in x$

Definition 2.4. $A \subseteq B$ denotes the formula $\forall z.z \in A \rightarrow z \in B$.

Exercise 1. *(1)* $A \subseteq A$.
(2) $\emptyset \subseteq A$.
(3) $A = B \leftrightarrow A \subseteq B \wedge B \subseteq A$
(4) $A \subseteq B \subseteq C \rightarrow A \subseteq C$

We cannot prove much more using $Ax1, Ax2, Ax3$ since $\{\emptyset\}$ is a model of $Ax1, Ax2, Ax3$.

2.2. **Pairing, Union and Replacement.** Using the following axioms, we can ensure that some of the most basic concepts in set theory exists and in particular prove the existence of non-empty sets.

$$\textbf{Ax4.(Pairing)}\ \forall x \forall y \exists w . x \in w \wedge y \in w$$

So using comprehension we can now prove the existence of the set $\{x, y\}$ and the set $\{x\}$ by applying paring to x, x and by a suitable comprehension. So we can now define order pairs:

Definition 2.5. the ordered pair of x and y is defined by

$$\langle x, y \rangle := \{\{x\}, \{x, y\}\}$$

Ordered pairs exists by applying pairing and comprehension to $\{x\}, \{x, y\}$.

Theorem 2.6 (Pairs equality)**.** *For every* x, y, x', y' *we have*

$$\langle x, y \rangle = \langle x', y' \rangle \leftrightarrow x = x' \wedge y = y'$$

Ax5.(Union) $\forall \mathcal{F} \exists Y.(\forall x.(\exists z.z \in \mathcal{F} \wedge x \in z) \rightarrow x \in Y)$

The set Y only includes the union of the sets in $\mathcal{F}$ but with comprehension we my form the (unique) set:

Definition 2.7. For any set $\mathcal{F}$ we define

$$\cup \mathcal{F} := \{x \in Y \mid \exists z \in \mathcal{F}.x \in z\}$$

where Y is the set guaranteed from Ax5.

Exercise 2. *Prove that the definition of $\cup\mathcal{F}$ does not depend on the choice of Y. Namely, if Y, Y' are two sets witnessing the union axiom for $\mathcal{F}$, then the resulting definition $\cup\mathcal{F}$ is the same.*

The following definition does not require the union axiom:

Definition 2.8. Let $\mathcal{F} \neq \emptyset$, define the intersection

$$\cap \mathcal{F} := \{x \mid \forall z \in \mathcal{F}.x \in z\}$$

Note that the intersection exists by comprehension and since it equal

$$\cap \mathcal{F} := \{x \in B \mid \forall z \in \mathcal{F}.x \in z\}$$

where B is any member of $\mathcal{F}$.

Exercise 3. *Define (and prove the existence) of the following objects: $A \cup B$, $A \cap B$, $A \setminus B$, $A \Delta B$.*

For every formula $\phi(x, y)$ such that Y is not free in $\phi(x, y)$, the universal closure of the following formula is an axiom:

Ax6.(Replacement scheme) $(\forall x \in A.\exists! y.\phi(x, y)) \rightarrow (\exists Y.\forall x \in A.\exists y \in Y.\phi(x, y))$

The intuition is that ϕ defines some function f using parameters with $x \in A$ as input and y as output. Then we can find a set Y such that for every $x \in A$, $f(x) \in Y$. Now using comprehension we can define the set

$$\{f(x) \mid x \in A\} := \{y \in Y \mid \exists x \in A.\phi(x, y)\}$$

Define the cartesian product of two sets as:

$$A \times B := \{\langle a, b\rangle \mid a \in A, b \in B\}$$

How do we justify the existence of this set?

- consider the formula $\phi(x, y) =$ "$y = \langle a, x\rangle$. Then for every a (the universal closure) we have that foe very $b \in B$, there is a unique y such that $\phi(b, y)$. So by replacement (and comprehension) we can define the set $\{a\} \times B$
- Now we use replacement again with the formula $\psi(x, y) =$ "$y = \{x\} \times B$" or formally[1]:

$$\psi(x, y) = \text{“}\forall z.z \in y \leftrightarrow \exists b \in B.z = \langle x, b\rangle\text{”}$$

[1]We leave the notation of $\langle a, b\rangle$ but it can be removed as well.

we see (by extensionality) that for every $x \in A$ there is a unique y such that $\psi(x, y)$ so we may form the set $\{\{x\} \times B \mid x \in A\}$.
- Finally, we define

$$A \times B = \cup\{\{x\} \times B \mid x \in A\}$$

Axioms Ax0,Ax1,A3-Ax6 suffice to develop the theory of relations, function, equivalence relations and orders. We assume that the reader is familiar with those definitions and we encourage her to construct all the standard objects from the axioms. The assumed theory and notations are given in the preliminaries chapter. We refer the reader to the first chapter of K.Kunen's book "introduction to independence proofs" for a complete account of the assumed material.

Remark 2.9. Note that the Powerset axiom is not needed in order to formulate this part of set theory and comes only later.

3. Well ordering

Recall that a (strong) order on a set A is a relation R which is transitive, reflexive, and strongly-anti-symmetric. R is total if every any two members $a, b \in A$ are R-comparable, namely: $a = b \vee aRb \vee bRa$.

Definition 3.1. An total order R on A is called a *well-order* if:

$$\forall X \subseteq A. X \neq \emptyset \Rightarrow \exists \min{}_R(X)$$

where $\min_R(X)$ is a (unique) element in $x \in X$ such that $\forall y \in X. x \neq y \Rightarrow xRy$.

Example 3.2.
- Every total order on a finite set is a well-order.
- $\mathbb{N}$ with the regular order is a well order.
- $\mathbb{N} \times \mathbb{N}$ with the lexicographic order is a well order.
- Consider the following order of ${}^{\mathbb{N}}\mathbb{N}$ given by fRg iff $f(n^*) < g(n^*)$ where $n^* = \min\{n \mid f(n) \neq g(n)\}$. Then R is a total ordering of ${}^{\mathbb{N}}\mathbb{N}$ which is not a well-order.

Definition 3.3. Let $\langle A, R\rangle, \langle B, S\rangle$ be ordered sets. An order-isomorphism between them is a bijection $f : A \to B$ which is order-preserving, namely: $\forall a, b \in A. aRb \Leftrightarrow f(a)Sf(b)$. We say that $\langle A, R\rangle \simeq \langle B, S\rangle$ if there is an isomorphism between them.

Definition 3.4. Let $\langle A, R\rangle$ be an ordered set, define $A_R[x] = \{y \in A \mid yRx\}$.

Lemma 3.5. *If $\langle A, R\rangle$ is a well order then for any $x \in A$, $\langle A, R\rangle \not\simeq \langle A_R[x], R\rangle$.*

Proof. Suppose that $f : R \to A_R[x]$ witnesses otherwise, let $B = \{y \mid f(y)Ry\}$. B is not empty since $f(x) \in A_R[x]$ and therefore $f(x)Rx$. Let $x^* = \min_R(B)$, then $f(x^*)Rx^*$ and since f is order preserving $f(f(x^*))Rf(x^*)$, hence $f(x^*) \in B$, contradictiong the minimality of x^*. □

Exercise 4. *Find a counter-example for the previous lemma in case that $\langle A, R\rangle$ is not well ordered.*

Lemma 3.6. *Suppose $\langle A, R\rangle, \langle B, S\rangle$ are well-orders and $\langle A, R\rangle \simeq \langle B, S\rangle$. Then the isomorphism between them is unique.*

Proof. Suppose that g_1, g_2 are two isomorphisms and toward contradiction assume that $g_1 \neq g_2$. Let $x_* = \min\{x \in A \mid g_1(x) \neq g_2(x)\}$. Then $g_1(x^*) \neq g_2(x^*)$. Without loss of generality, suppose taht $b := g_1(x^*) S g_2(x^*)$ and let yRx^* be such that $g_2(y) = b$, then $g_1(y) S g_1(x^*) = b = g_2(y)$, thus $g_1(y) \neq g_2(y)$ and therefore $y \in \{x \mid g_1(x) \neq g_2(x)\}$ contradiction the minimality of x^*. □

Definition 3.7. Let $\langle A, R\rangle$ be a well-ordering A set $X \subseteq A$ is called an initial segment if $\forall y \in X \forall z \in A. zRy \to z \in X$.

Lemma 3.8. *Let $\langle A, R\rangle$ be a well-ordering and $X \subseteq A$. Then X is an initial segment iff $X = A$ or $\exists x \in A. A_R[x] = X$.*

Proof. Exercise. [Hint: define $x = \min A \setminus X$] □

Theorem 3.9 (The trichotomy theorem of well-ordering)**.** *Let $\langle A, R\rangle$,$\langle B, S\rangle$ be well-ordering. Then exactly one of the following holds:*

(1) $\langle A, R\rangle \simeq \langle B, S\rangle$.
(2) there is $x \in A$ such that $\langle A_R[x], R\rangle \simeq \langle B, S\rangle$.
(3) there is $y \in B$ such that $\langle A, R\rangle \simeq \langle B_S[y], S\rangle$.

Proof. Let

$$f = \{\langle a, b\rangle \in A \times B \mid \langle A_R[a], R\rangle \simeq \langle B_S[b], S\rangle\}$$

First we claim that $dom(f), Im(f)$ are initial segments. To see this, is suffices to prove that they are downward closed. For example, if $a'Ra$ and $a \in \text{dom}(f)$ then there is b such that $\langle A_R[a], R\rangle \simeq \langle B_S[b], S\rangle$. Let $g : A_R[a] \to B_S[b]$ be an isomorphism witnessing this. Note that $A_R[a']$ is an initial segment of $A_R[a]$ and therefore $g \restriction A_R[a']$ is defined, order preserving and $1-1$. Let $b' = g(a')$, it is not hard to verify that $Im(g) = B_S[b']$ and therefore $g \restriction A_R[a']$ witnesses the fact that $\langle A_R[a'], R\rangle \simeq \langle B_S[b'], S\rangle$ which implies that $a' \in dom(f)$. Similarily, $Im(f)$ is an initial segment. Also f must be (univalent and) injective since otherwise, we would have had $a_1 R a_2$ such that $b = f(a_1) = f(a_2)$ and in particular $\langle A_R[a_1], R\rangle \simeq \langle B_S[b], S\rangle \simeq \langle A_R[a_2], R\rangle$ which contradicts the lemma that a well ordering is not isomorphic to its proper initial segments.

Finally, we claim that it is impossible that both $dom(f), Im(f)$ are proper initial segment, sense otherwise, $dom(f) = A_R[x]$ and $Im(f) = B_S[y]$ and we let $x' = \min A \setminus A_R[x]$ and $y' = \min B \setminus B_S[y]$, then we can extend f to be defined on $A_R[x']$ by sending $f(x) = y$ witnessing that $x' \in dom(f)$, contradiction. □

4. ORDINALS

The basic theory is due to Von Neuman.

Definition 4.1. A set x is called trastivie if

$$\forall y \in x \forall z \in y . z \in x \equiv \forall y \in x . y \subseteq x$$

Example 4.2. $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$

Exercise 5. *If $\mathcal{F}$ is a set of transitive sets then $\cup\mathcal{F}, \cap\mathcal{F}$ are both transitive sets.*

Transitive sets are sets for which the $\in$-relation is transitive.

Definition 4.3. A set α is called an ordinal if α is a transitive set and

$$\in_\alpha := \{\langle x, y\rangle \in \alpha^2 \mid x \in y\}$$

is a well order on α.

Remark 4.4. The axiom of foundation and the axiom of choice will later tell us that an infinite decreasing $\in$- sequence does not exist and therefore it will suffice to require that $\in_\alpha$ is a total order.

Example 4.5. $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ the set $\{\emptyset, \{\emptyset\} \{\{\emptyset\}\}\}$ is an example of a transitive set which is not an ordinal (since $\emptyset$ and $\{\{\emptyset\}\}$ are not $\in$-comperable). If $x = \{x\}$ then x is not an ordinal since we will have $x \in x$ and therefore $\in$ is not anti reflexive. For the same reason, for every ordinal α, $\alpha \notin \alpha$.

Theorem 4.6.
1. *If α is an ordinal and $x \in \alpha$ then x is an ordinal and $x = \alpha_\in[x]$.*
2. *$\alpha \subseteq \beta$ iff $\alpha \in \beta \vee \alpha = \beta$.*
3. *If α, β are ordinals such that $\alpha \simeq \beta$ then $\alpha = \beta$.*
4. *For every two ordinal α, β, $\alpha \in \beta \vee \beta \in \alpha \vee \alpha = \beta$.*
5. *If C is a set of ordinals then there is $\min_\in(C)$.*
6. *If C is a set of ordinals then $\cup C$ is an ordinal and had the property of supremum, namely, it is an upper bound of C: $\forall \alpha \in C . \alpha \subseteq \cup C$ and if β is an upper bound for C then $\cup C \subseteq \beta$.*

Proof. (1) exercise. (2), from right to left is easy. From left to right, suppose that $\alpha \subseteq \beta$ and $\alpha \neq \beta$, let $\gamma = \min(\beta \setminus \alpha)$, we claim that $\gamma = \alpha$. If $x \in \gamma$, then $x \in \beta$ and bu minimality of γ, $x \in \alpha$. If $x \in \alpha$, then $x \in \beta$ by inclusion. x, γ are comparable in $\in$, but $\gamma = x$ and $\gamma \in x$ is ruled out since $\gamma \in \beta \setminus \alpha$, so $x \in \gamma$. By double inclusion $\alpha = \gamma$. For (3), suppose that there is $x \in \alpha$ such that $f(x) \neq x$ and let x be the minimal such x. Then x is an ordinal and $x = f[x] \subseteq \beta$, but then $x \in \beta$ and $x \in f(x)$ so there is $y \in \alpha$ such that $f(y) = x \neq y$ but then $y \in x$ since f is order-preserving which contradicts the minimality of x. (4) follows from (1), (3) and the trichotomy theorem □

Corollary 4.7. $\neg\exists z . \forall x . x$ *is an ordinal* $\Rightarrow x \in z$

Proof. Otherwise, let $On = \{\alpha \in z \mid \alpha \text{ is an ordinal}\}$ (which exists by comprehansion), then On is a transitive set (by (1) of the previous theorem) and $\in$ well orders On (by (3) and (5)) and therefore On is itself an ordinal, so $On \in On$. However, no ordinal can be a member of itself, contradiction. □

We denote the class of all ordinals by On.

Remark 4.8. As we have just proved, there is no formal object which is On in the mathematical universe, thus, there is no formal distinction between "$x \in On$" and "x is an ordinal", or "$A \subset On$" and "$\forall x \in A$, x is an ordinal".

Theorem 4.9. *For any well-ordered set $\langle A, R\rangle$ there is a unique ordinal α such that $\langle A, R\rangle \simeq \langle \alpha, \in\rangle$. We call this α the order-type of $\langle A, R\rangle$ and denote it by* $\text{otp}(A, R)$.

Proof. Uniqueness follows from before. To prove existence, let $B = \{a \in A \mid \exists x \in On.\langle A_R[a], R\rangle \simeq \langle x, \in\rangle\}$. Note that for every $a \in B$, there is a unique ordinal x which witness $a \in B$. So we may apply replacement to B and form the set $C = \{x \in On \mid \exists a \in B.\langle A_R[a], R\rangle \simeq \langle x, \in\rangle\}$. We claim that C is an ordinal. First, since C is a set of ordinal, the $\in$ relation on C is a well order. To see that C is transitive, note that if $y \in x \in C$ and $\langle A_R[a], R\rangle \simeq \langle x, \in\rangle$ then there is $b \in A_R[a]$ such that $\langle A_R[b], R\rangle \simeq \langle y, \in\rangle$. Hence $b \in B$ and $y \in C$. It follows that C is an ordinal. A similar argument proves that B is an initial segment of A and if $B = A_R[c]$ for some c then $c \in B$ by definition so $B = A$. □

Remark 4.10. Without the axiom of replacement, one cannot prove theorem 4.8 as there is a model of $ZFC - \{Ax6\}$ for which theorem 4.8 fails.

Notation 4.11. $\alpha < \beta$ iff $\alpha \in \beta$ and $\alpha \leq \beta$ iff $\alpha < \beta \vee \alpha = \beta$ iff $\alpha \subseteq \beta$.

Theorem 4.12.
(1) *If α is an ordinal then $\emptyset \leq \alpha$.*
(2) *If α is an ordinal then $\alpha + 1 := \alpha \cup \{\alpha\}$ is an ordinal and is the successor of α in the sense that it is the minimal ordinal greater than α.*
(3) *If A is a set of ordinals without a greatest element then* $\sup A := \cup A$ *is an ordinal strictly greater then all the ordinals in A.*

Proof. Exercise. □

Definition 4.13. A successor ordinal is an ordinal of the form $\alpha + 1$, otherwise it is called limit.

Definition 4.14. Let $0 = \emptyset$ and recursively define $n + 1 = n \cup \{n\}$.

Definition 4.15. α is anatural number iff $\forall \beta \leq \alpha.\beta = 0 \vee \beta$ is successor.

How can we argue that the set of natural numbers exists? Well we cannot just from the current axioms

Ax7.(Infinity) $\exists x.(0 \in X \wedge \forall y \in x.y \cup \{y\} \in x)$

By induction, x contains all the natural numbers (Formally: otherwise, suppose that n is a natural number such that $n \notin X$. Then $n \neq 0$ and thus $n = m \cup \{m\}$ for some natural $m < n$. Let $n' = \min(n \setminus X)$, then $n' = m' \cup \{m'\}$ and $m' < n'$ so $m' \in X$ and therefore $n' \in X$, contradiction.) Now by comprehension we can define the set of natural numbers:

Definition 4.16. Denote by ω or $\mathbb{N}$, the set of all natural numbers.

ω is the first limit ordinal above 0. Note that we do not care if ω is "really" the set natural numbers, and all we care about is that it is some realization of the Peano postulates:

Theorem 4.17. *Let $S(n) = n + 1$, then $\langle \omega, \emptyset, S\rangle$ is a model of Peano postulates:*

(1) $0 \in \omega$.
(2) $n \in \omega \to S(n) \in \omega$.
(3) $n \neq m \Rightarrow S(n) \neq S(m)$.
(4) *(induction)* $\forall X \subseteq \omega.(0 \in X \wedge n \in X \Rightarrow n + 1 \in X) \Rightarrow X = \omega$

5. TRANSFINITE RECURSION AND INDUCTION

We will formulate the induction and recursion theorem in a way that can be applied to what we call classes. Formally, a class does not exist as a mathematical object (as we have seen for V and for On). Given a formula $\pi(x)$ with a free variable x (we allow other free variables, indeed, the class we are defining might depend on parameters) we think of the class C_ϕ as the "collection" (whatever that means) $C_\phi = \{x \mid \phi(x)\}$. So whenever C_ϕ appears in a mathematical statement, it should be clear how to replace C_ϕ by ϕ, for example:

(1) $\forall x \in C_\phi.x$ satisfy... just mean $\forall x.\phi(x) \Rightarrow x$ satisfy...
(2) $C_\phi \subseteq On$ means

$$\forall x.\phi(x) \Rightarrow x \text{ is an ordinal.}$$

Note that if C_ϕ is a class and A is a set then $C_\phi \cap A = \{x \in A \mid \phi(x)\}$ which is a set that exists by comprehansion.

The next theorems are formulated for classes and take their usual meaning when the class is in fact a set:

Theorem 5.1. *Let $0 \neq C$ be a class of ordinal (formally, let ϕ be a formula such that $(\exists x.\phi(x)) \wedge (\forall x.\phi(x) \Rightarrow x$ is an ordinal)). Then there is $y = \min(C)$ (formally, $\exists y.\phi(y) \wedge \forall x.\phi(x) \to x \geq y$).*

Proof. Let $\alpha \in C$ be any ordinal, then $D = \alpha + 1 \cap C$ is a non-empty set of ordinals, and therefore $y = \min(D)$ exists. Let us prove that $y = \min(C)$. let $x \in C$, then either $x > \alpha$ in which chase $x > \alpha \geq y$ of $x \leq y$ but then $x \in D$ and therefore $x \geq y$. $\square$

Formally, what we have above is a theorem scheme, one for every formula ϕ. This theorem enables us to prove the induction theorem over all the ordinal!:

Theorem 5.2 (The induction theorem)**.** *Let Ψ be any formula.*

$$(\forall\alpha \in On.[\forall\beta < \alpha.\Psi(\beta)] \Rightarrow \Psi(\alpha)) \Rightarrow \forall\alpha \in On.\Psi(\alpha)$$

Proof. Suppose otherwise, we let C be the class of all ordinals α such that $\neq \Psi(\alpha)$. By our assumption $0 \neq C \subseteq On$. Apply the previous theorem to find $\alpha^* = \min(C)$, clearly, for every $\beta < \alpha^*$, $\Psi(\beta)$ should hold. However $\neg\Psi(\alpha^*)$ holds, which contradicts the assumption of the theorem. □

Theorem 5.3 (The recursion theorem)**.** *Suppose that $F(x, y)$ is a formula such that $\forall x\exists!y.F(x, y)$. Then one can write down a formula $G(v, w)$ such that*

$$\forall\alpha \in On.\exists!w.G(\alpha, w) \wedge \forall\alpha \in On\exists x.\exists y.(x = G \restriction \alpha \wedge F(x, y) \wedge G(\alpha, y)$$

Before proving the theorem, let us explain the formulation of the theorem. The formula $F(x, y)$ is thought of as the formula $f(x) = y$ for some "function" $f : V \to V$ which accommodates some recursive information. Then the theorem says that there is a function $g : On \to V$ (which is given by the formula $G(v, w)$) such that for every $\alpha \in On$, $g(\alpha) = f(g \restriction \alpha))$.

To see how this relates to the usual way we define functions recursively, recall that in a recursive definition of a function, we assume that $\forall\beta < \alpha$, $g(\beta)$ has already been defined (in other words, $g \restriction \alpha$ has been defined) and given this unknown definition we define $g(\alpha)$. The purpose of the function f is to take that unknown $x = g \restriction \alpha$, which can be have any possible values, and the output $g(x)$ is what we would have wanted for the value of $g(\alpha)$ to be. The recursion theorem simply tells you that given a function f (which is defined on any possible sequence x) the function g which satisfies $g(\alpha) = f(g\restriction\alpha)$ exists. Since we are talking about classes, this is all formulated with formulas instead of functions.

Proof. the statement "g is δ-approximation" means that "g is a function and $\text{dom}(g) = \delta$ and for every $\rho < \delta$, $F(g\restriction\rho, g(\rho))$". We prove by transfinite induction that for each δ, there is a unique δ-approximation. Then the formula $G(v, w)$ is

$$G(v, w) \equiv \exists\nu < \delta.\exists g.g \text{ is a } \delta\text{-approximation} \wedge g(\nu) = w$$

The for every ordinal α, since a δ approximation is unique, there is a unique w such that $G(\alpha, w)$. Also, since we will prove that δ-approximations exists, the second part of the formula will be satisfied. □

Remark 5.4. In many situations we use the induction and recursion theorem simultaneously when we define a function g and assume that $g\restriction\alpha$ has already been defined and satisfies some properties, then we define $g(\alpha)$ and prove it satisfies some properties.

Example 5.5. Ordinal arithmetic: for a fixed α, we define:

- $\alpha + \beta$ by recursion on β
  (1) $\alpha + 0 = \alpha$.
  (2) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$.
  (3) For a limit ordinal δ, we define $\alpha + \delta = \sup_{\beta<\delta} \alpha + \beta$.
- $\alpha \cdot \beta$ by recursion on β
  (1) $\alpha \cdot 0 = \alpha$.
  (2) $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$.
  (3) For a limit ordinal δ, we define $\alpha \cdot \delta = \sup_{\beta<\delta} \alpha \cdot \beta$.
- α^β by recursion on β
  (1) $\alpha^0 = 1$.
  (2) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$.
  (3) For a limit ordinal δ, we define $\alpha^\delta = \sup_{\beta<\delta} \alpha^\beta$.

$1 + \omega = \sup_{n<\omega} 1 + n = \omega < \omega_1$

$2 \cdot \omega = \sup_{n<\omega} 2 \cdot n = \omega < \omega + \omega = \omega + 2$
$2^\omega = \sup_{n<\omega} 2^n = \omega$ (so 2^ω as ordinals and as cardinal is not the same!)
$\omega + \omega^2 = \omega^2$
$(\omega + 1)^2 = (\omega + 1) \cdot (\omega + 1) = (\omega + 1) \cdot \omega + \omega + 1 = \omega^2 + \omega + 1.$

Proposition 5.6. *(1) If $\alpha < \beta$ then for every γ, $\gamma + \alpha < \gamma + \beta$.*
(2) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

Proof. For (1), We prove by trasfinite induction of β that for every $\alpha < \beta$ and every γ, $\gamma + \alpha < \gamma + \beta$. For $\beta = 0$, the claim is vacuously true (since there is no $\alpha < 0$). Suppose that the claim holds for β and let us prove it for $\beta + 1$. Let $\alpha < \beta + 1$ and γ be any ordinal. Let us split into cases:

- If $\alpha < \beta$, then by the induction hypothesis and the definition of "+" in the successor case,

$$\gamma + \alpha < \gamma + \beta < (\gamma + \beta) + 1 = \gamma + (\beta + 1)$$

- If $\alpha = \beta$, then as in the first case we get $\gamma + \beta < \gamma + (\beta + 1)$.

For limit β, let $\alpha < \beta$, then $\alpha + 1 < \beta$. By the induction hypothesis applies to $\alpha + 1$ and the definition of "+" is the limit case,

$$\gamma + \alpha < \gamma + (\alpha + 1) \leq \sup_{\delta<\beta} \alpha + \delta = \alpha + \beta$$

For (2), agan we prove it by induction on γ, for every α, β.

- For $\gamma = 0$ we have that:

$$(\alpha + \beta) + 0 = \alpha + \beta = \alpha + (\beta + 0)$$

- At successor step $\gamma + 1$, we have that

$$(\alpha+\beta)+(\gamma+1) = ((\alpha+\beta)+\gamma)+1 = (\alpha+(\beta+\gamma))+1 = \alpha+((\beta+\gamma)+1) = \alpha+(\beta+(\gamma+1))$$

- At limit steps γ, suppose that for every $\delta < \gamma$ we have that $(\alpha+\beta)+\delta = \alpha + (\beta + \delta)$, then

$$(\alpha + \beta) + \gamma = \sup_{\delta<\gamma}(\alpha + \beta) + \delta = \sup_{\delta<\beta} \alpha + (\beta + \delta) =^{*} \alpha + (\beta + \gamma)$$

To see why $*$ holds, we will use (1) and the definition of supremum. Indeed, if $\delta < \gamma$ then from (1) we get that $\beta + \delta < \beta + \gamma$ and therefore (again from (1)), $\alpha+(\beta+\delta) < \alpha+(\beta+\gamma)$. Hence $\sup_{\delta<\gamma} \alpha+(\beta+\delta) \leq \alpha+(\beta+\gamma)$. Note that $\beta+\gamma = \sup_{\delta<\gamma} \beta+\delta$ by definition and therefore (since $\beta+\delta$ is strictly incresing with δ) we conclude that $\beta+\gamma$ is a limit ordinal and that $\sup\{\alpha + \rho \mid \rho < \beta + \gamma\}$. It follows that

$$\alpha + (\beta + \gamma) = \sup_{\rho<\beta+\gamma} \alpha + \rho$$

Hence we need to check that

$$\sup_{\delta<\gamma} \alpha + (\beta + \delta) = \sup_{\rho<\beta+\gamma} \alpha + \rho$$

We have that $\{\beta+\delta \mid \delta < \gamma\} \subseteq \beta+\gamma$ so " $\leq$ " is clear (the sup is taken over more elements). For the other direction, let $\rho < \beta + \gamma$ then there is $\delta < \gamma$ such that $\beta+\delta > \rho$ and by (1) we have that $\alpha+(\beta+\delta) > \alpha+\rho$ so " $\geq$ " follows.

□

6. Cardinals

Definition 6.1. Let A, B be any sets. We say that:
(1) $A \sim B$ if there is $f : A \to B$ which is invertible.
(2) $A \precsim B$ if there is $f : A \to B$ which is injective.
(3) $A \prec B$ if $A \precsim B$ and $A \not\sim B$.

Theorem 6.2 (Cantor-Berstein)**.** *Let A, B be sets and supose that $A \precsim B \wedge B \precsim A$ then $A \sim B$.*

If A can be well ordered, then there α such that $A \sim \alpha$

Definition 6.3. Suppose that A can be well ordered. Denote by $|A|$ to be the minimal ordinal α such that $A \sim \alpha$.

Definition 6.4. An ordinal α is called a cardinal if $\alpha = |\alpha|$. Equivalently, if for every $\beta < \alpha$, $\beta < |\alpha|$.

Note that $|A|$ is only defined for sets which can be well ordered. We will further discuss this problem in the subsection about the axiom of choice.

Exercise 6. *(1) If $|\alpha| \leq \beta \leq \alpha$ then $|\alpha| = |\beta|$.*
(2) $n \not\sim n + 1$ for every n. [Hint: induction.]
(3) If $|\alpha| = n$ then $\alpha = n$.

Corollary 6.5. *ω is a cardinal and every $n \in \omega$ is a cardinal.*

Proof. Otherwise, $|\omega| < \omega$ and therefore $|\omega| = n$ so there $|\omega| < n+1 < \omega$, but then $|n+1| = |\omega| = n$, contradicting the $n \not\sim n+1$. □

Definition 6.6. A is finite if there is n such that $|A| = n$. A is countable if $|A| = \omega$. A is uncountable if $|A| > \omega$.

Proposition 6.7. *The following sets are countable:* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N}\times\mathbb{N}, \mathbb{N}^n$ *for* $n \in \mathbb{N}_+, \cup_{n<\omega}\mathbb{N}^n$.

Based on the axioms so far, one cannot prove that there is an infinite set A such that $|A| > \omega$ (or even $A \not\sim \omega$). We can still say the following:

Lemma 6.8. *Every infinite cardinal is a limit ordinal.*

Proof. Otherwise, $\kappa = \alpha + 1$. and since κ is infinite (ordinal), then $\kappa \geq \omega$. It is not hard to construct a bijection between $\alpha + 1$ and α in this situation and therefore $|\kappa| \leq \alpha < \kappa$, contradiction. □

Ax8.(Power set) $\forall x \exists y. \forall z. z \subseteq x \Rightarrow z \in y$

Definition 6.9. Let $P(x) = \{b \mid b \subseteq x\}$.

Theorem 6.10 (Cantor's theorem)**.** *For every set* x, $x \prec P(x)$.

Theorem 6.11. $P(x) \sim {}^x\{0,1\}$

Proposition 6.12. *If* $A \sim A'$ *then* $P(A) \sim P(A')$.

So we now have sets which are not countable. But what about uncountable sets? the problem is that $P(\omega)$ might not admit a well order.

Theorem 6.13. *For every ordinal* α *there is a cardinal* κ *such that* $\alpha < \kappa$.

Proof. Suppose otherwise, that there is an ordinal α such that for every cardinal κ is at most α. In particular, for every ordinal $\beta \geq \alpha$, $\beta \sim \alpha$. Let $S = \{R \in P(\alpha \times \alpha) \mid R \text{ well-orders } \alpha\}$. S exists by the power set axiom and comprehansion. Define for each $R \in S$, $F(R) = \text{otp}(\alpha, R)$. Then by replacement the following is a set exists $\{E = F(R) \mid R \in S\}$. By our assumption, for every $\beta \geq |\alpha|$, $\beta \sim \alpha$, and we can translate the order $(\beta, \in)$ to a well order R on α such that $\text{otp}(\alpha, R) = \beta$. We conclude that $E = \{\beta \in On \mid \beta \geq |\alpha|\}$. This is a contradiction to the fact that On was already proven not to be a set (Show that the set E cannot be a set!). □

Definition 6.14. For every α, denote by α^+ the minimal cardinal $\alpha < \kappa$. a cardinal of the form α^+ is called a successor cardinal and a cardinal κ such that for every $\alpha < \kappa$, $\alpha^+ < \kappa$ is called a limit cardinal.

Definition 6.15 (The $\aleph$ hierarchy)**.** By transfinite recursion we define $\aleph_\alpha$ for every ordinal $\alpha \in On$. $\omega_0 = \aleph_0 := \omega$ $\omega_{\alpha+1} = \aleph_{\alpha+1} := \aleph_\alpha^+$ and for a limit δ, $\omega_\delta = \aleph_\delta := \sup_{\alpha<\delta} \aleph_\alpha$.

Theorem 6.16. *(1) Every* $\aleph_\alpha$ *is a cardinal*

(2) For every infinite cardinal κ, there is α such that $\aleph_\alpha = \kappa$.
(3) If $\alpha < \beta$ then $\aleph_\alpha < \aleph_\beta$.
(4) $\aleph_\alpha$ is limit cardinal iff α is a limit ordinal and $\aleph_\alpha$ is a successor cardinal iff α is a successor ordinal.

Proof. For (1), we go by induction of α, the base case and succesoor case are easy by the definition of $\aleph_{\alpha+1}$. For limit δ, suppose toward a contradiction that $|\aleph_\delta| < \aleph_\delta$, then by definition of sup, there is $\alpha < \delta$ such that $|\aleph_\delta| < \aleph_\alpha$. Since δ is limit, we have that $\alpha + 1 < \delta$ and therefore

$$\aleph_\alpha < \aleph_{\alpha+1} \leq \aleph_\delta$$

Which implies by previous exercises that $|\aleph_\alpha| = |\aleph_\delta| < \aleph_\alpha$, contradicting the fact that $\aleph_\alpha$ is a cardinal by the induction hypothesis. As for (2), let κ be a cardinal and let $\delta = \sup\{\gamma \mid \aleph_\gamma \leq \kappa\}$. We claim that $\aleph_\delta = \kappa$. Let us split into cases: if $\delta = \max(\{\gamma \mid \aleph_\gamma \leq \kappa\})$, then $\aleph_\delta \leq \kappa$ and by maximality $\aleph_{\delta+1} = \aleph_\delta^+ > \kappa$. It follows that $\kappa = |\kappa| = \aleph_\delta$. If δ is limit, then again, since $\aleph_\delta = \sup_{\alpha<\delta} \aleph_\alpha$, it follows that $\aleph_\delta \leq \kappa$. It follows again that $\aleph_\delta^+ > \kappa$ and thus $\aleph_\delta = |\kappa| = \kappa$. (3) and (4) are left as exercises. □

7. The real numbers

It is possible to define $\langle \mathbb{Z}, +, \cdot, < \rangle$ (similar to field of fractions but with $+$) and $\mathbb{Q}$ from pure algebraic constructions from $\mathbb{N}$ (the field of fractions). But it is not clear how to define $\mathbb{R}$. The set theoretic approach is to use order/topological properties to characterize $\mathbb{R}$. Before moving to the definition of $\mathbb{R}$, let us prove that $\mathbb{Q}$ also has an order characterization:

Theorem 7.1 (Cantor's theorem)**.** *Let $\langle A, <_A \rangle$ be an ordered set such that:*
(1) $|A| = \aleph_0$.
(2) $\langle A, <_A \rangle$ has no least and last element.
(3) A is dense in itself, namely for every $a_1, a_2 \in A$, if $a_1 <_A a_2$ then there is $a_3 \in A$ such that $a_1 <_A a_3$ and $a_3 <_A a_2$.

In term of Logic, this is to say the theory of dense ordered set without first and last element is $\aleph_0$-catgorical.

There are two usual ways to realize $\mathbb{R}$, either with Cauchy sequences or with Dedekind cuts. We will follow the latter.

Definition 7.2. A set $X \subseteq \mathbb{Q}$ is called a Dedekind cut if $X \neq \emptyset, \mathbb{Q}$, X has no maximal element, and X is downward closed, namely, $\forall x \in X \forall y \in \mathbb{Q}. y < x \Rightarrow y \in X$.

$$\mathbb{R} := \{X \in P(\mathbb{Q}) \mid X \text{ is a Dedekind cut}\}$$

We order Dedekind cuts by inclusion. There is a standard way to identify $\mathbb{Q}$ inside $\mathbb{R}$, by $q \mapsto \mathbb{Q}_<[q]$. This function is $1-1$ and order-preserving.

Example 7.3. The set $X = \{q \in \mathbb{Q} \mid q < 0 \vee q^2 < 2\}$ is a Dedekind cut and there is no $q \in \mathbb{Q}$ such that $X = \mathbb{Q}_<[q]$

Theorem 7.4. *$\mathbb{Q}$ is dense in $\mathbb{R}$*

Proof. If $X_1 < X_2$ are any cuts, fix any $q \in X_2 \setminus X_1$ then $X_1 \leq q < X_2$. Since X_2 has no maximal element, there is $q' \in X_2$ such that $q < q'$, then clearly, $X_1 < q' < X_2$. □

Definition 7.5. A ordered set $\langle A, R\rangle$ is complete if any bounded set X has a least upper bound (supremum)

Theorem 7.6. *$\mathbb{R}$ is complete*

Proof. Let $\mathcal{F} \subseteq \mathbb{R}$ be a non empty bounded set of reals, then $\cup\mathcal{F}$ is a Dedekind cut which is the supremum of $\mathcal{F}$. □

Theorem 7.7. *$\mathbb{R}$ is the unique (up to isomorphism) ordered $\langle A, R\rangle$ set such that:*
(1) $\langle A, R\rangle$ has no first and last element.
(2) $\langle A, R\rangle$ contains a countable dense subset. (separability)
(3) $\langle A, R\rangle$ is complete.

The cardinality of $\mathbb{R}$ is called the continuum and is denoted by $\mathfrak{c}$.

Theorem 7.8. *$\mathbb{R}$ is not countable*

Proof. Suppose otherwise, then $\mathbb{R} = \{r_n \mid n \in \mathbb{N}\}$. Let us define a sequence:

$$a_0 < a_1 < a_2 ... a_n < ... < b_n < b_{n-1} ... < b_2 < b_1 < b_0$$

as follows: $a_0 = r_0$ and $b_0 = r_k$ for the minimal k such that $r_k > a_0$. Suppose that $a_n < b_n$ were defined and let $a_{n+1} = r_k$ for the minimal k such that $a_n < r_k < b_n$. Let $b_{n+1} = r_k$ for the minimal k such that $a_{n+1} < r_k < b_n$. By the completeness of $\mathbb{R}$ there is $a = \sup_{n<\omega} a_n$. Note that for every n, $a_n < a < b_n$. There is k^* such that $a = r_{k^*}$ and there if $l > k^*$ such that for some n, $b_n = r_l$. This means that at stage $n-1$, we had $a_{n-1} < b_{n-1}$ and we chose $b_n = r_k$ for the minimal k such that $a_{n-1} < r_k < b_{n-1}$ and this minimal k was l. However, $a_{n-1} < a = r_{k^*} < b_{n-1}$ also satisfies this property and $k^* < l$, contradiction. □

Theorem 7.9. $\mathbb{R} \sim P(\mathbb{N})$

Theorem 7.10. $|(\alpha, \beta)| = |[\alpha, \beta]| = |\mathbb{R}|$

7.1. Three questions about the real numbers.

Question 1. *Can the real numbers be well-ordered?*

Question 2. *The continuum hypothesis: is there a set $A \subseteq \mathbb{R}$ such that $A \not\sim 0, 1, 2, ..., \omega, \mathbb{R}$?*

Definition 7.11. An ordered set $\langle A, R\rangle$ is called CCC (countable chain condition) if whenever I is a set of disjoint open intervals in A, then $|I| \leq \aleph_0$.

Proposition 7.12. *$\mathbb{R}$ is ccc.*

Question 3. *Suslin hypothesis: If we replace separability by ccc do we still obtain a characterization of $\mathbb{R}$*

8. The axiom of choice

Every time we perform the following:

$$"X \neq \emptyset \text{ let } x \in X"$$

we are making a choice. This line can appear only finitely many times in a formal proof and therefore we are allowed to choose finitely many times. However, we encounter a problem if we would like to choose infinitely many times and we need to introduce the axiom of choice:

Ax9.(Choice)$\forall \mathcal{A}.(\forall a \in \mathcal{A}.a \neq \emptyset) \Rightarrow (\exists f : \mathcal{A} \to \cup\mathcal{A}.\forall a \in \mathcal{A}.f(a) \in a)$

We denote the axiom of choice by AC. Here are some basic theorems which use the axiom of choice:
(1) If $g : A \to B$ is onto then there is $f : B \to A$ such that $g \circ f = Id_B$.
(2) If A is infinite then $\mathbb{N} \precsim A$.
(3) $A \precsim B$ iff there is a function $f : B \to A$ which is onto.
(4) The countable union of countable sets is countable.

Other non set-theoretic examples:
(1) Every field has an algebraically closed closure.
(2) Every ideal is contained in a maximal ideal.
(3) There exists a set which is not Lebesgue measurable.
(4) Tychonoff's theorem: a product of compact topological spaces is compact.
(5) Hahn-Banach theorem.
(6) Completeness theorem for first order logic.
(7) the compactness theorem for first order logic.
(8) $\mathbb{R}$ can be well ordered.

Theorem 8.1. *The following are equivalent:*
- *AC.*
- *Every set can be well ordered.*
- *Zorn's lemma*

Corollary 8.2 (AC)**.** *For every set A, $|A|$ is well defined and in particular the cardinals form a class of representatives for all possible cardinalities.*

Corollary 8.3 (AC)**.** *The cardinalities of all sets are well ordered.*

Corollary 8.4 (AC)**.** *For any set A, $|A \times A| = |A|$.*

Definition 8.5 (AC)**.** The Continuum Hypothesis (CH) is the statement that $|\mathbb{R}| = \aleph_1$.

9. cardinal arithmetic

Definition 9.1. Let κ, λ be cardinals. we define:
(1) $\kappa + \lambda = |\kappa \times \{0\} \uplus \lambda \times \{1\}|$.
(2) $\kappa \cdot \lambda = |\kappa \times \lambda|$.
(3) $\kappa^\lambda = |{}^\lambda\kappa|$

Exercise 7. *If* $|A| = |A'|$ *and* $|B| = |B'|$ *then:*
(1) $|A \times \{0\} \uplus B \times \{1\}| = |A' \times \{0\} \uplus B' \times \{1\}|$.
(2) $|A \times B| = |A' \times B'|$.
(3) $|{}^B A| = |{}^{B'} A'|$

Theorem 9.2 (Basic properties- Not assuming AC)**.** *Let* κ, λ, σ *be any cardinals (finite or infinite) then*
(1) $\kappa + \lambda = \lambda + \kappa$, $\kappa \cdot \lambda = \lambda \cdot \kappa$ *(commutativity)*
(2) $(\kappa + \lambda) + \sigma = \kappa + (\lambda + \sigma)$, $\kappa \cdot (\lambda \cdot \sigma) = (\kappa \cdot \lambda) \cdot \sigma$.*(Associativity)*
(3) $\kappa \cdot (\lambda + \sigma) = \kappa \cdot \lambda + \kappa \cdot \sigma$.*(Distributively)*
(4) $\kappa + 0 = \kappa$, $\kappa \cdot 0 = 0$, $\kappa \cdot 1 = \kappa$, $\kappa^1 = \kappa$, $1^\kappa = 1$, $0^0 = 1$, *for* $\kappa > 0$, $0^\kappa = 0$. *(Neutral elements)*
(5) *For every* n $\underbrace{\kappa + \kappa + \kappa + \kappa + \ldots + \kappa}_{n\ times} = n \cdot \kappa$, $\underbrace{\kappa \cdot \kappa \cdot \kappa \cdot \kappa \cdot \ldots \cdot \kappa}_{n\ times} = \kappa^n$.

Monotonicity: If $\kappa \leq \lambda$ *and* $\sigma \leq \tau$ *then*
(1) $\kappa + \sigma \leq \lambda + \tau$.
(2) $\kappa \cdot \sigma \leq \lambda \cdot \tau$.
(3) $\kappa^\sigma \leq \lambda^\tau$.

Rules of exponent:
(1) $(\kappa^\lambda)^\sigma = \kappa^{\lambda \cdot \sigma}$.
(2) $\kappa^{\lambda + \sigma} = \kappa^\lambda \cdot \kappa^\sigma$.
(3) $(\kappa \cdot \lambda)^\sigma = \kappa^\sigma \cdot \lambda^\sigma$

Note that for natural numbers this is the usual definition of addition, multiplication and power.

Notation 9.3. $\kappa^{<\lambda} = \sup_{\delta < \lambda} \kappa^\delta$.

Corollary 9.4 (AC)**.** *If* κ, λ *are infinite then:*
(1) $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$.
(2) *Suppose that for every* $\alpha < \kappa$, X_α *is a set such that* $|X_\alpha| \leq \kappa$. *Then* $|\cup_{\alpha < \kappa} X_\alpha| \leq \kappa$
(3) *For* $\delta \leq \kappa$, $\kappa^\delta = |[\kappa]^\delta|$ *where* $\kappa^\delta = \{X \in P(\kappa) \mid |X| = \delta\}$.
(4) $\kappa^{<\omega} = \kappa$.

Proof. For (2), for each $\alpha < \kappa$ choose a function $f_\alpha : \kappa \to X_\alpha$ which is onto. Then define a function $f : \kappa \times \kappa \to \cup_{\alpha < \kappa} X_\alpha$ by $f(\alpha, \beta) = f_\alpha(\beta)$. Then f is onto and therefore $|\cup_{\alpha<\kappa} X_\alpha| \leq \kappa \cdot \kappa = \kappa$. For (3), The function $F(f) = Im(f)$ is an onto fnction from ${}^\delta\kappa$ to $[\kappa]^\delta$. For the other direction, ${}^\delta\kappa \subseteq \{R \in P(\kappa \times \delta) \mid |R| = \delta\}$. Since $|P(\kappa \times \delta)| = |P(\kappa)$ we get that $|{}^\delta\kappa| \leq |[\kappa]^\delta|$. For (4), note that $\kappa^n = \kappa$ for every $n \geq 1$ (by induction and since $\kappa \cdot \kappa = \kappa$) and therefore $\kappa^{<\omega} = \sup_{n<\omega} \kappa^n = \kappa$ □

It follows that $\kappa^{<\delta} = |[\kappa]^{<\delta}|$ where $[A]^{<\delta} = \{B \subseteq A \mid |B| < \delta\}$. Also from (1) we see that only the exponent operation is left unsettled. As we will see later, ZFC cannot determine theses values. However, there are some cases which are settled, in the rest of this chapter we investigate what restrictions ZFC pose one these values:

Theorem 9.5. *If $\lambda \geq \omega$ and $2 \leq \kappa \leq \lambda$ then*

$$\kappa^\lambda = 2^\lambda$$

Proof. $2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^\lambda$. □

In case $\lambda < \kappa$ we can say a bit more about κ^λ but we need the following definition:

Definition 9.6. Let α be an **ordinal**. We define $cf(\alpha)$ to be the minimal γ such that there is an cofinal/unbounded function $f : \gamma \to \alpha$[2].

Example 9.7. $cf(\omega) = \omega$, $cf(\omega_1) = \omega_1$ (since if $\alpha < \omega_1$ and $f : \alpha \to \omega_1$, we have that $\sup(f)$ is a countable union of countable sets so $|\sup(f)| = \omega$. It follows that $\sup(f) < \omega_1$.

Remark 9.8. (1) $cf(\alpha) \leq \alpha$.
(2) $cf(\alpha + 1) = 1$.
(3) there is always $f : cf(\alpha) \to \alpha$ which is cofinal and strictly increasing.

Exercise 8. *If α is a limit ordinal and $f : \alpha \to \beta$ is cofinal and strictly increasing then $cf(\alpha) = cf(\beta)$.*

Exercise 9. *For every limit ordinal α, $cf(\aleph_\alpha) = \alpha$.*

Corollary 9.9. *$cf(cf(\beta)) = cf(\beta)$.*

Definition 9.10. a limit ordinal κ called *regular* if $cf(\kappa) = \kappa$, otherwise it is called *singular*.

Corollary 9.11. *If κ is regular then κ is a cardinal.*

Example 9.12. ω is regular and ω_1 is regular. $cf(\aleph_\omega) = \omega < \aleph_\omega$ is singular.

Theorem 9.13 (AC)**.** *For every κ, κ^+ is regular.*

Proof. Otherwise, there is a function $f : \lambda \to \kappa^+$ for some $\lambda \leq \kappa$. For every $\alpha < \lambda$, let $X_\alpha = f(\alpha)$, then $|X_\alpha| \leq \kappa$ and therefore $|\kappa^+| = |\cup_{\alpha<\lambda} X_\alpha| \leq \kappa$, contradiction. □

Is there a limit regular cardinal greater than $\aleph_0$?

Definition 9.14. A cardinal κ is called
(1) Weakly inaccessible if it regular and a limit cardinal.
(2) Strongly inaccessible if it is regular and

$$\forall\lambda < \kappa.2^\lambda < \kappa$$

weakly and strongly inaccessible cardinals are so-called "large cardinals", these are cardinals which ZFC cannot prove their existence.

Lemma 9.15 (Konig's Lemma)**.** *Let κ be an infinite cardinal, and assume that $cf(\kappa) \leq \lambda$, then $\kappa^\lambda > \kappa$*

[2] $f : \gamma \to \alpha$ is cofinal/unbounded if $Im(f)$ is inbounded in α.

Proof. Let $f : \lambda \to \kappa$ be cofinal. Suppose toward a contradiction that there is $G : \kappa \to {}^{\lambda}\kappa$ which is onto. Define $g : \lambda \to \kappa$ by

$$g(\alpha) = \min(\kappa \setminus \{G(\mu)(\alpha) \mid \mu < f(\alpha)\})$$

To see that $g \notin Im(G)$, let $\rho < \kappa$ then there is $\beta < \lambda$ such that $\rho < f(\beta)$. Hence $g(\beta) \notin \{G(\mu)(\beta) \mid \mu < f(\beta)\}$ and in particular $g(\beta) \neq G(\rho)(\beta)$, hence $g \neq G(\rho)$. This is a contradiction to the fact he G is onto. □

Corollary 9.16. *For any infinite cardinal κ, $cf(2^\kappa) > \kappa$.*

Proof. Note that $(2^\kappa)^\kappa = \kappa$, hence by the contrapositive of Konig's lemma, we get $cf(2^\kappa) > \kappa$. □

9.1. **The continuum function.** The function $\alpha \mapsto 2^{\aleph_\alpha}$ is called the continuum function and as we will see, its values are highly undetermined by ZFC.

Definition 9.17 (AC)**.** The Generalized Continuum Hypothesis (GCH) is the statement that for every α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Under GCH, all the values of κ^λ (and therefore the continuum function) can easily be computed,

Theorem 9.18 (AC+GCH)**.** *Let λ, κ be infinite cardinals. Then:*
(1) If $\lambda \geq \kappa$, then $\kappa^\lambda = \lambda^+$.
(2) If $cf(\kappa) \leq \lambda < \kappa$ then $\kappa^\lambda = \kappa^+$.
(3) If $\lambda < cf(\lambda)$ then $\kappa^\lambda = \kappa$

Proof. It remains to prove 3, so $\kappa \leq \kappa^\lambda = \sup_{\delta<\kappa} \delta^\lambda \leq \sup_{\delta<\kappa} \delta^+ = \kappa$ □

Let us define the *beth function*:

Definition 9.19. $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ and for limit δ, $\beth_\delta = \sup_{\alpha<\delta} \beth_\alpha$.

Exercise 10. *GCH is equivalent to the statement that for every α, $\beth_\alpha = \aleph_\alpha$.*

To summarize what we know about the continuum function, we have the following theorem:

Theorem 9.20. *(1) $\kappa < \lambda \Rightarrow 2^\kappa \leq 2^\lambda$. (Monotonicity)*
(2) $cf(2^\kappa) > \kappa$. (Konig's lemma)
(3) If κ is limit then $2^\kappa = (2^{<\kappa})^{cf(\kappa)}$.

Proof. We need to prove (3), $\kappa = \sup_{i<cf(\kappa)} \kappa_i$. So the map

$$X \subseteq \kappa \mapsto \langle X \cap \kappa_i \mid i < cf(\kappa)\rangle$$

is a $1-1$ function from $P(\kappa)$ to ${}^{cf(\kappa)}([\kappa]^{<\kappa})$. Hence

$$2^\kappa \leq (2^{<\kappa})^{cf(\kappa)} \leq (2^\kappa)^{cf(\kappa)} = 2^{\kappa \cdot cf(\kappa)} = 2^\kappa$$

□

In case κ is regular, (3) is not very interesting and as we will see, constrains (1), (2) are the only limitations ZFC pose on the continuum function in ZFC. However, (3), suggests that for singular cardinals the situation is very different and depends heavily on thecontinuum function restricted to cardinals below it and on the exponent values. For example we have the following corollary:

Corollary 9.21. *If κ is singular, and the continuum function is eventually constant below κ with value λ, then $2^\kappa = \lambda$.*

Definition 9.22. A cardinal κ is strong limit if $\forall \nu < \kappa . 2^\nu < \kappa$.

Note that a strong limit cardinal is in particular a limit cardinal.

Exercise 11. *(1) Prove that there is a strong limit cardinal and that the least such carinal is of cofinality ω.*
(2) Prove that if κ is strong limit then:

$$\forall \nu, \lambda < \kappa . \lambda^\nu < \kappa$$

(3) If κ is strong limit then $2^\kappa = \kappa^{cf(\kappa)}$

Definition 9.23. The *Singular Cardinal Hypothesis* is the statement:

For every strong limit singular cardinal κ, $2^\kappa = \kappa^+$

There is another formulation which implies the above, which involves all singular cardinals:

For every singular cardinal κ, $2^{cf(\kappa)} < \kappa \Rightarrow \kappa^{cf(\kappa)} = \kappa^+$

We will leave it as an exercise to prove that the second formulation determines the continuum function for all singular cardinals. While the second version implies the first, it is known that the two formulations are not equivalent.

10. Infinite combinatorics

Many finite combinatorial principles generalize to the infinite. For example:

Theorem 10.1 (Pigeonhole Principle)**.** *If $\kappa < cf(\lambda)$ then for every function $f : \lambda \to \kappa$ there is $i < \kappa$ such that $|f^{-1}[\{i\}]| = \lambda$.*

Proof. Otherwise, $\lambda = \cup_{i<\kappa} f^{-1}[\{i\}]$ and each $\theta_i := |f^{-1}[\{i\}]| < \lambda$ then let $\theta = \sup_{i<\kappa} \theta_i < \lambda$ but then $\lambda = |\cup_{i<\kappa} f^{-1}[\{i\}]| \leq \max(\kappa, \theta) < \lambda$, contradiction. □

Theorem 10.2 (Ramsey Theorem)**.** *Let $f : [\omega]^2 \to n$ be any function (coloring of the full graph with n-many colors) then there is $H \subseteq \omega$, $|H| = \omega$ such that $f[H]^2$ is constant.*

Example 10.3. This does not generalize to higher cardinals: There is a function $G : [2^\kappa]^2 \to \kappa$ with no homogenous set of size 3.

one of the most important tools in infinite combinatorics is ”clubs” and ”stationery sets”.

10.1. **Clubs and stationery sets.**

Definition 10.4. Let κ be a limit ordinal $cf(\kappa) > \omega$. A set $C \subseteq \kappa$ is call closed if for every $\alpha < \kappa$, if $\sup(C \cap \alpha) = \alpha$ then $\alpha \in C$. C is called a *club* if it is closed and unbounded.

Example 10.5. (1) Sets of the form $(\alpha, \kappa) = \{\beta < \kappa \mid \alpha < \beta\}$ are clubs.
(2) $\{\alpha < \kappa \mid \alpha \text{ is a limit ordinal}\}$ is a club.
(3) $\{\alpha + 1 \mid \alpha < \kappa\}$ is not a club.
(4) If A is unbounded then $cl(A) := \{\alpha < \kappa \mid \sup(A \cap \alpha) = \alpha\}$ is a club.
(5) $\{\omega^\alpha \mid \alpha < \kappa\}$ is a club.
(6) If C is a club and $f : \kappa \to \kappa$ is continuous and increasing then $f[C]$ is a club.

Clubs are in some sense ”large” subsets of κ.

Definition 10.6. Let $f : [\kappa]^n \to \kappa$ be any function. A closure point of f is some $\alpha < \kappa$ such that $f''[\alpha]^n \subseteq \alpha$.

Exercise 12. *κ be regular, then for every function $f : [\kappa]^n \to \kappa$, then set $C_f = \{\alpha < \kappa \mid \alpha$ is a closure point of $f\}$ is a club.*

Proof. To see that C_f is closed, suppose that $\alpha = \sup(C_f \cap \alpha)$. In particular, α is a limit point. Let us prove that $\alpha \in C_f$, if $\vec{\beta} \in [\alpha]^n$, then there is $\alpha' \in C_f \cap \alpha$ such that $\vec{\beta} \in [\alpha']^n$ and since $\alpha' \in C_f$, then $f(\vec{\beta}) < \alpha' < \alpha$. To see that C_f is unbounded, let $\delta < \kappa$, define a sequence $\langle \alpha_k \mid k < \omega\rangle$ recursively: $\alpha_0 = \delta + 1$ and $\alpha_{k+1} = \sup(f''[\alpha_k]^n)$. Let $\alpha^* = \sup_{k<\omega} \alpha_k$, then $\delta < \alpha^* \in C_f$ since if $\vec{\beta} \in [\alpha^*]^n$, then there is $k < \omega$ such that $\vec{\beta} \in [\alpha_k]^n$ and therefore $f(\vec{\beta}) < \alpha_{k+1} < \alpha^*$. □

Proposition 10.7. *Suppose that κ is an ordinal of uncountable cofinality, the intersection of less than $cf(\kappa)$-many clubs is a club*

Proof. Let $\langle C_i \mid i < \lambda\rangle$ be clubs and $cf(\kappa) > \lambda$. Let us prove that $\cap_{i<\lambda} C_i$ is a club. It is straightforward to see that the intersection of closed sets is closed. To see that it is unbounded, let $\delta < \kappa$ and let us construct a sequence $\langle \alpha_n \mid n < \omega\rangle$ as follows: $\alpha_0 = \delta$. Suppose that α_k was defined and let us defined α_{k+1} as the limit of the sequence $\langle \beta^{k+1}_j \mid j < \lambda\rangle$ where β_j is defined again recursively as follows:

$$\beta_0 \in C_0 \setminus \alpha_k$$

and

$$\beta_j \in C_j \setminus \sup_{i<j} \beta_i$$

Note that $\sup_{i<j} \beta_i < \kappa$ by the cofinality assumption. Let $\alpha^* = \sup_{k<\omega} \alpha_k$ and let us prove that $\alpha^* \in \cap_{j<\lambda} C_j$. Indeed for every $j < \lambda$, we have

$\alpha_k < \beta_j^{k+1} < \alpha_{k+1}$ and therefore $\alpha^* = \sup_{k<\omega} \beta_j^{k+1}$ so $\sup(C_j \cap \alpha^*) = \alpha^*$ and since C_j is a club, $\alpha^* \in C_j$. □

Definition 10.8. Let $\langle A_i \mid i < \kappa \rangle$ be a sequence of subsets of κ. Define

(0) $$\Delta_{i<\kappa} A_i := \{\alpha < \kappa \mid \forall \beta < \alpha . \beta \in A_\alpha$$

Example 10.9. $\Delta_{i<\kappa}(i, \kappa) = \kappa$, $\Delta_{i<\kappa}(i+1, \kappa) = Lim(\kappa)$.

Proposition 10.10. *For every $i < \kappa$, $(\Delta_{j<\kappa} A_j) \setminus i+1 \subseteq A_i$.*

Theorem 10.11. *Let $\langle C_i \mid i < \kappa \rangle$ be a sequence of clubs, then $\Delta_{i<\kappa} C_i$ is a club.*

Proof. Let C^* denote the diagonal intersection. To see that it is closed, let $\alpha = \sup(C^* \cap \alpha)$, and let $i < \alpha$, then for every $\beta \in (i, \alpha) \cap C^*$ we have that $i < \beta \ni C^*$ and so $\beta \in C_i$. it follows that $\alpha = \sup(\alpha \cap C_i)$ and therefore $\alpha \in C_i$. To see it is unbounded, let us construct a sequence $\langle \alpha_n \mid n < \omega \rangle$: $\alpha_0 = \delta$, $\alpha_{n+1} \in \cap_{i<\alpha_n} C_i \setminus \alpha_n + 1$ and $\alpha^* = \sup_{n<\omega} \alpha_n$. To see that $\alpha^* \in C^*$, let $i < \alpha^*$, then there is $n < \omega$ such that $\alpha_n > i$ and for each $m \geq n$, $\alpha_m \in C_i$ so $\alpha^* = \sup(C^* \cap \alpha^*)$, and $\alpha^* \in C^*$. □

Definition 10.12. A subset $S \subset \kappa$ is called stationary if $S \cap C \neq \emptyset$ for every club C.

Example 10.13. (1) Clubs are stationery sets.
(2) S is non-stationary iff $S \subseteq \kappa \setminus C$ for some club C. $\{\alpha + 1 \mid \alpha < \kappa\}$ is non-stationary
(3) $C \cap S$ is stationary.
(4) $\{\alpha < \kappa \mid cf(\alpha) = \lambda\} =: E^\kappa_\lambda$ is stationary.

Theorem 10.14 (Fodor's theorem)**.** *Let S be stationary and $f : S \to \kappa$ be regressive. Then there is a stationary set $S' \subseteq S$ such that $f \restriction S'$ is constant.*

Proof. Suppose not, then for every $i < \kappa$, $f^{-1}[\{i\}]$ is no stationary, and therefore the is a club C_i such that $C_i \cap f^{-1}[\{i\}] = \emptyset$. Let $C = \Delta_{i<\kappa} C_i$, then C is a club and therefore $C \cap S \neq$. Let $\alpha \in C \cap S$ then $i = f(\alpha) < \alpha$ since f is regressive. Thus $\alpha \in f^{-1}[\{i\}]$. but since $\alpha \in C \setminus i+1$, we have that $\alpha \in C_i$ contradiction. □

Theorem 10.15. *The Δ-system lemma.*

Theorem 10.16 (Ulam)**.** *there are κ-many pairwise disjoint stationary subsets of κ.*